

# MEAN CURVATURE FLOW VIA CONVEX FUNCTIONS ON GRASSMANNIAN MANIFOLDS

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ABSTRACT. Using the convex functions in Grassmannian manifolds we can carry out interior estimates for mean curvature flow of higher codimension. In this way some of the results in [5] can be generalized to higher codimension

## 1. INTRODUCTION

We consider the deformation of a complete submanifold in  $\mathbb{R}^{m+n}$  under the mean curvature flow. For codimension one case there are many deep results given by Ecker-Huisken [4][5][7] and [8].

In recent years some interesting work has been done for higher codimensional mean curvature flow [1][2][3][9][10][11][12] and [13]. In a previous paper the first author studied mean curvature flow with convex Gauss image [17]. Some results in [4] has been generalized to higher codimensional situation. The present work would carry out interior estimates and generalize some results in [5] to higher codimension.

For a hypersurface there are support functions which play an important role in gradient estimates for mean curvature flow of codimension one. For general submanifolds we can also define generalized support functions related to the generalized Gauss map whose image is the Grassmannian manifold. The Plücker imbedding of the Grassmannian manifold into Euclidean space gives us the "height functions"  $w$  on the Grassmannian manifold. In the case of positive "height function" we can give lower bound of the Hessian of  $\frac{1}{w}$  in our previous paper [18]. Based on it we can define auxiliary functions which enable us to carry out gradient estimates for MCF in higher codimension from which we obtain confinable properties (Theorem 4.1) as well as curvature estimates (Theorem 5.1 and Theorem 5.2). In this way, we improve the previous results in [17].

## 2. CONVEX FUNCTIONS ON GRASSMANNIAN MANIFOLDS

Let  $\mathbb{R}^{m+n}$  be an  $(m+n)$ -dimensional Euclidean space. All oriented  $n$ -subspaces constitute the Grassmannian manifolds  $\mathbf{G}_{n,m}$ .

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Fix  $P_0 \in \mathbf{G}_{n,m}$  in the sequel, which is spanned by a unit  $n$ -vector  $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ . For any  $P \in \mathbf{G}_{n,m}$ , spanned by an  $n$ -vector  $e_1 \wedge \cdots \wedge e_n$ , we define an important function on  $\mathbf{G}_{n,m}$ ,

$$w \stackrel{\text{def.}}{=} \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle = \det W,$$

where  $W = (\langle e_i, \varepsilon_j \rangle)$ .

Denote

$$\mathbb{U} = \{P \in \mathbf{G}_{n,m} : w(P) > 0\}.$$

Let  $\{\varepsilon_{n+\alpha}\}$  be  $m$  vectors such that  $\{\varepsilon_i, \varepsilon_{n+\alpha}\}$  form an orthonormal basis of  $\mathbb{R}^{m+n}$ . Then we can span arbitrary  $P \in \mathbb{U}$  by  $n$  vectors  $f_i$ :

$$f_i = \varepsilon_i + z_{i\alpha} \varepsilon_{n+\alpha},$$

where  $Z = (z_{i\alpha})$  are the local coordinates of  $P$  in  $\mathbb{U}$ . Here and in the sequel we use the summation convention and agree the range of indices:

$$1 \leq i, j \leq n; \quad 1 \leq \alpha, \beta \leq m.$$

The Jordan angles between  $P$  and  $P_0$  are defined by

$$\theta_\alpha = \arccos(\lambda_\alpha),$$

where  $\lambda_\alpha \geq 0$  and  $\lambda_\alpha^2$  are the eigenvalues of the symmetric matrix  $W^T W$ . On  $\mathbb{U}$  we can define

$$v = w^{-1}.$$

Then it is easily seen that

$$v(P) = [\det(I_n + ZZ^T)]^{\frac{1}{2}} = \prod_{\alpha=1}^m \sec \theta_\alpha.$$

The canonical metric on  $\mathbf{G}_{n,m}$  in the local coordinates can be described as (see [15] Ch. VII)

$$(2.1) \quad g = \text{tr}((I_n + ZZ^T)^{-1} dZ (I_m + Z^T Z)^{-1} dZ^T).$$

Let  $E_{i\alpha}$  be the matrix with 1 in the intersection of row  $i$  and column  $\alpha$  and 0 otherwise. Denote  $g_{i\alpha, j\beta} = \langle E_{i\alpha}, E_{j\beta} \rangle$  and let  $(g^{i\alpha, j\beta})$  be the inverse matrix of  $(g_{i\alpha, j\beta})$ . Then,

$$(1 + \lambda_i^2)^{\frac{1}{2}} (1 + \lambda_\alpha^2)^{\frac{1}{2}} E_{i\alpha}$$

form an orthonormal basis of  $T_P \mathbf{G}_{n,m}$ , where  $\lambda_\alpha = \tan \theta_\alpha$ . Denote its dual basis in  $T_P^* \mathbf{G}_{n,m}$  by  $\omega_{i\alpha}$ .

A lengthy computation yields [18]

$$\begin{aligned}
 \text{Hess}(v)_P = & \sum_{m+1 \leq i \leq n, \alpha} v \omega_{i\alpha}^2 + \sum_{\alpha} (1 + \lambda_{\alpha}^2) v \omega_{\alpha\alpha}^2 + v^{-1} dv \otimes dv \\
 (2.2) \quad & + \sum_{\alpha < \beta} \left[ (1 + \lambda_{\alpha} \lambda_{\beta}) v \left( \frac{\sqrt{2}}{2} (\omega_{\alpha\beta} + \omega_{\beta\alpha}) \right)^2 \right. \\
 & \left. + (1 - \lambda_{\alpha} \lambda_{\beta}) v \left( \frac{\sqrt{2}}{2} (\omega_{\alpha\beta} - \omega_{\beta\alpha}) \right)^2 \right].
 \end{aligned}$$

Define

$$B_{JX}(P_0) = \left\{ P \in \mathbb{U} : \begin{array}{l} \text{sum of any two Jordan angles} \\ \text{between } P \text{ and } P_0 < \frac{\pi}{2} \end{array} \right\}.$$

This is a geodesic convex set, larger than the geodesic ball of radius  $\frac{\sqrt{2}}{4}\pi$  and centered at  $P_0$ . This was found in a previous work of Jost-Xin [6]. For any real number  $a$  let  $\mathbb{V}_a = \{P \in \mathbf{G}_{n,m}, \quad v(P) < a\}$ . From ([6], Theorem 3.2) we know that

$$\mathbb{V}_2 \subset B_{JX} \quad \text{and} \quad \overline{\mathbb{V}}_2 \cap \overline{B}_{JX} \neq \emptyset$$

$\text{Hess}(v)_P$  is positive definite if and only if  $\theta_{\alpha} + \theta_{\beta} < \frac{\pi}{2}$  for arbitrary  $\alpha \neq \beta$ , i.e.,  $P \in B_{JX}(P_0)$ .

From (2.2) it is easy to get an estimate

$$\text{Hess}(v) \geq v(2-v)g + v^{-1}dv \otimes dv \quad \text{on } \overline{\mathbb{V}}_2.$$

For later applications the above estimate is not accurate enough. Using the radial compensation technique the estimate could be refined.

**Theorem 2.1.** [18]

$v$  is a convex function on  $B_{JX}(P_0) \subset \mathbb{U} \subset \mathbf{G}_{n,m}$ , and

$$\text{Hess}(v) \geq v(2-v)g + \left( \frac{v-1}{pv(v^{\frac{2}{p}}-1)} + \frac{p+1}{pv} \right) dv \otimes dv$$

on  $\overline{\mathbb{V}}_2$ , where  $g$  is the metric tensor on  $\mathbf{G}_{n,m}$  and  $p = \min(n, m)$ .

**Remark 2.1.** For any  $a \leq 2$ , the sub-level set  $\mathbb{V}_a$  is a convex set in  $\mathbf{G}_{n,m}$ .

**Remark 2.2.** The sectional curvature varies in  $[0, 2]$  under the canonical Riemannian metric. By the standard Hessian comparison theorem we have

$$\text{Hess}(\rho) \geq \sqrt{2} \cot(\sqrt{2}\rho)(g - d\rho \otimes d\rho),$$

where  $\rho$  is the distance function from a fixed point in  $\mathbf{G}_{n,m}$ .

## 3. EVOLUTION EQUATIONS

Let  $M$  be a complete  $n$ -submanifold in  $\mathbb{R}^{m+n}$ . Consider the deformation of  $M$  under the mean curvature flow, i.e.  $\exists$  a one-parameter family  $F_t = F(\cdot, t)$  of immersions  $F_t : M \rightarrow \mathbb{R}^{m+n}$  with corresponding images  $M_t = F_t(M)$  such that

$$(3.1) \quad \begin{aligned} \frac{d}{dt} F(x, t) &= H(x, t), \quad x \in M \\ F(x, 0) &= F(x), \end{aligned}$$

where  $H(x, t)$  is the mean curvature vector of  $M_t$  at  $F(x, t)$ .

From equation (3.1) it is easily known that

$$(3.2) \quad \left( \frac{d}{dt} - \Delta \right) |F|^2 = -2n.$$

Let  $B$  denote the second fundamental form of  $M_t$  in  $\mathbb{R}^{m+n}$ . It satisfies the evolution equation

**Lemma 3.1.** (*Lemma 3.1 in [17]*)

$$(3.3) \quad \left( \frac{d}{dt} - \Delta \right) |B|^2 \leq -2|\nabla|B||^2 + 3|B|^4.$$

The Gauss map  $\gamma : M \rightarrow \mathbf{G}_{n,m}$  is defined by

$$\gamma(x) = T_x M \in \mathbf{G}_{n,m}$$

via the parallel translation in  $\mathbb{R}^{m+n}$  for  $\forall x \in M$ . The Gauss maps under the MCF satisfies the following relation.

**Proposition 3.1.** [13]

$$(3.4) \quad \frac{d\gamma}{dt} = \tau(\gamma(t)),$$

where  $\tau(\gamma(t))$  is the tension fields of the Gauss map from  $M_t$ .

Let  $h : \mathbb{V} \rightarrow \mathbb{R}$  be a smooth function defined on an open subset  $\mathbb{V} \subset G_{n,m}$  and denote  $\tilde{h} = h \circ \gamma$ , then

$$\frac{d\tilde{h}}{dt} = \frac{d(h \circ \gamma)}{dt} = dh(\tau(\gamma)).$$

On the other hand, by the composition formula

$$\Delta \tilde{h} = \Delta(h \circ \gamma) = \text{Hess}(h)(\gamma_* e_i, \gamma_* e_i) \circ \gamma + dh(\tau(\gamma)),$$

where  $\{e_i\}$  is a local orthonormal frame field on  $M_t$ ; and then we derive

$$(3.5) \quad \left( \frac{d}{dt} - \Delta \right) \tilde{h} = -\text{Hess}(h)(\gamma_* e_i, \gamma_* e_i) \circ \gamma.$$

## 4. CONFINABLE PROPERTIES

Now, we consider the convex Gauss image situation which is preserved under the flow, so called confinable property.

Let  $r : \mathbb{R}^{n+m} \times \mathbb{R} \rightarrow \mathbb{R}$  be a smooth, nonnegative function, such that for any  $R > 0$ ,

$$\overline{M}_{t,R} = \{x \in M_t : r(x, t) \leq R^2\}$$

is compact.

**Lemma 4.1.** *Assume  $r$  satisfies  $(\frac{d}{dt} - \Delta)r \geq 0$ . Let  $R > 0$ , such that  $\gamma(\overline{M}_{0,R}) \subset \mathbb{V} \subset \mathbf{G}_{n,m}$ . Define  $\varphi = R^2 - r$  and  $\varphi_+$  denotes the positive part of  $\varphi$ .  $h : \mathbb{V} \rightarrow \mathbb{R}$  is a smooth positive function such that*

$$(4.1) \quad \text{Hess}(h) \geq Ch^{-1}dh \otimes dh$$

with  $C \geq \frac{3}{2}$ . Then we have the estimate

$$\tilde{h}\varphi_+^2 \leq \sup_{\overline{M}_{0,R}} \tilde{h}\varphi_+^2,$$

where  $\tilde{h} = h \circ \gamma$ .

*Proof.* Denote  $\eta = \varphi_+^2$ , then at an arbitrary interior point of the support of  $\varphi_+$ , we have

$$(4.2) \quad \eta' \leq 0, \quad \eta^{-1}(\eta')^2 = 4, \quad \text{and } \eta'' = 2,$$

where  $'$  denotes differentiation with respect to  $r$ . By (4.1), (3.5), we have

$$(4.3) \quad \left(\frac{d}{dt} - \Delta\right)\tilde{h} \leq -C\tilde{h}^{-1}|\nabla\tilde{h}|^2$$

and moreover

$$\begin{aligned} & \left(\frac{d}{dt} - \Delta\right)(\tilde{h}\eta) \\ &= \left(\frac{d}{dt} - \Delta\right)\tilde{h} \cdot \eta + \tilde{h}\left(\frac{d}{dt} - \Delta\right)\eta - 2\nabla\tilde{h} \cdot \nabla\eta \\ &\leq -C\tilde{h}^{-1}|\nabla\tilde{h}|^2\eta + \tilde{h}\left(\eta'\left(\frac{d}{dt} - \Delta\right)r - \eta''|\nabla r|^2\right) - 2\nabla\tilde{h} \cdot \nabla\eta \\ (4.4) \quad &\leq -C\tilde{h}^{-1}|\nabla\tilde{h}|^2\eta - 2\tilde{h}|\nabla r|^2 - 2\nabla\tilde{h} \cdot \nabla\eta. \end{aligned}$$

Observe that

$$\begin{aligned} -2\nabla\tilde{h} \cdot \nabla\eta &= (2C - 2)\nabla\tilde{h} \cdot \nabla\eta - 2C\nabla\tilde{h} \cdot \nabla\eta \\ &= (2C - 2)\eta^{-1}(\nabla(\tilde{h}\eta) - \tilde{h}\nabla\eta) \cdot \nabla\eta - 2C\nabla\tilde{h} \cdot \nabla\eta \\ &\leq (2C - 2)\eta^{-1}\nabla\eta \cdot \nabla(\tilde{h}\eta) - (2C - 2)\tilde{h}\eta^{-1}|\nabla\eta|^2 \\ &\quad + C\tilde{h}^{-1}|\nabla\tilde{h}|^2\eta + C\tilde{h}\eta^{-1}|\nabla\eta|^2 \\ (4.5) \quad &= (2C - 2)\eta^{-1}\nabla\eta \cdot \nabla(\tilde{h}\eta) + C\tilde{h}^{-1}|\nabla\tilde{h}|^2\eta + (8 - 4C)\tilde{h}|\nabla r|^2. \end{aligned}$$

Here (4.2) has been used. Substituting (4.5) into (4.4) gives

$$(4.6) \quad \left(\frac{d}{dt} - \Delta\right)(\tilde{h}\eta) \leq (2C - 2)\eta^{-1}\nabla\eta \cdot \nabla(\tilde{h}\eta) + (6 - 4C)\tilde{h}|\nabla r|^2$$

on the support of  $\varphi_+$ , The weak parabolic maximal principle then implies the result.  $\square$

**Lemma 4.2.** *Assume  $r$  satisfies  $(\frac{d}{dt} - \Delta)r \geq 0$ . If  $\gamma(M_t) \subset \mathbb{V}$  for arbitrary  $t \in [0, T]$  ( $T > 0$ ),  $h : \mathbb{V} \rightarrow \mathbb{R}$  is a smooth positive function satisfying (4.1) with  $C \geq 1$ , then for arbitrary  $a \geq 0$ , the following estimate holds.*

$$(4.7) \quad \sup_{M_t} \tilde{h}(1+r)^{-a} \leq \sup_{M_0} \tilde{h}(1+r)^{-a}.$$

*Proof.* By  $(\frac{d}{dt} - \Delta)r \geq 0$ ,

$$(4.8) \quad \begin{aligned} \left(\frac{d}{dt} - \Delta\right)(1+r)^{-a} &= -a(1+r)^{-a-1}\left(\frac{d}{dt} - \Delta\right)r - a(a+1)(1+r)^{-a-2}|\nabla r|^2 \\ &\leq -a(a+1)(1+r)^{-a-2}|\nabla r|^2. \end{aligned}$$

In conjunction with (4.3), we have

$$(4.9) \quad \begin{aligned} &\left(\frac{d}{dt} - \Delta\right)[\tilde{h}(1+r)^{-a}] \\ &\leq -C\tilde{h}^{-1}(1+r)^{-a}|\nabla\tilde{h}|^2 - a(a+1)\tilde{h}(1+r)^{-a-2}|\nabla r|^2 - 2\nabla\tilde{h} \cdot \nabla(1+r)^{-a} \\ &= -C\tilde{h}^{-1}(1+r)^{-a}|\nabla\tilde{h}|^2 - a(a+1)\tilde{h}(1+r)^{-a-2}|\nabla r|^2 + 2a\nabla\tilde{h} \cdot (1+r)^{-a-1}\nabla r. \end{aligned}$$

$C \geq 1$  implies  $Ca(a+1) \geq a^2$ , then by Young's inequality,

$$\left(\frac{d}{dt} - \Delta\right)[\tilde{h}(1+r)^{-a}] \leq 0.$$

Hence (4.7) follows from maximal principle for parabolic equations on complete manifolds (see [4]).  $\square$

**Theorem 4.1.** *If the initial submanifold is an entire graph over  $\mathbb{R}^n$ , i.e.,  $M_0 = \text{graph } f_0$ , where  $f_0 = (f_0^1, \dots, f_0^m)$ ,  $f_0^\alpha = f_0^\alpha(x^1, \dots, x^n)$ ; and*

$$\Delta_{f_0} < 2,$$

where

$$\Delta_f(x) = \left[ \det \left( \delta_{ij} + \frac{\partial f^\alpha}{\partial x^i}(x) \frac{\partial f^\alpha}{\partial x^j}(x) \right) \right]^{1/2}.$$

*Then the submanifolds under the MCF are still entire graphs over the same hyperplane, i.e.,  $M_t = \text{graph } f_t$ ; and*

$$\Delta_{f_t} < 2.$$

*Moreover, if  $(2 - \Delta_{f_0})^{-1}$  has growth*

$$(2 - \Delta_{f_0})^{-1}(x) \leq C_0(|x|^2 + 1)^a$$

where  $C_0, a$  are both positive constants, then the growth of  $(2 - \Delta_{f_t})^{-1}$  can be controlled by

$$(2 - \Delta_{f_t})^{-1} \leq 2C_0(|x|^2 + 2nt + 1)^a.$$

*Proof.* Define  $h = v^{\frac{3}{2}}(2 - v)^{-\frac{3}{2}}$ , then on  $\{P : v(P) < 2\}$ , we have (see [18], inequality (4.6))

$$\begin{aligned} \text{Hess}(h) &= h' \text{Hess}(v) + h'' dv \otimes dv \\ (4.10) \quad &\geq 3hg + \frac{3}{2}h^{-1}dh \otimes dh. \end{aligned}$$

Define  $r(x, t) = |F|^2 + 2nt$ , then  $(\frac{d}{dt} - \Delta)r = 0$ . Hence, the estimate in Lemma 4.1 holds. For arbitrary  $x_0 \in M_{t_0}$ , choose  $R > 0$ , such that  $r(x_0, t_0) < R^2$ , then  $\varphi_+(x_0, t_0) > 0$  and Lemma 4.1 implies

$$(4.11) \quad \tilde{h}(x_0, t_0) \leq \frac{1}{\varphi_+(x_0, t_0)} \sup_{\overline{M}_{0,R}} \tilde{h}\varphi_+^2 < +\infty.$$

Noting that  $\tilde{h} \rightarrow +\infty$  when  $v \rightarrow 2_-$  we have  $v(x_0, t_0) < 2$  and the first result follows. For  $\mathbf{x} \in \mathbb{R}^n$ , it is not difficult to see that

$$(\frac{d}{dt} - \Delta)(|\mathbf{x}|^2 + 2nt) \geq 0.$$

Now, we define  $r = |\mathbf{x}|^2 + 2nt$ , then the second assertion easily follows from Lemma 4.2.  $\square$

Choose

$$h = \sec^2(\sqrt{2}\rho)$$

and by the similar argument we can improve the previous result of the first author [17] as follows

**Theorem 4.2.** *If the Gauss image of the initial complete submanifold  $M_0$  is contained in an open geodesic ball of the radius  $R_0 \leq \frac{\sqrt{2}}{4}\pi$  in  $\mathbf{G}_{m,n}$ , then the Gauss images of all the submanifolds under the MCF are also contained in the same geodesic ball. Moreover, if*

$$(\frac{\sqrt{2}}{4}\pi - \rho)^{-1} \leq C_0(|F|^2 + 1)^a \quad \text{on } M_0,$$

(Here  $\rho$  denotes the distance function on  $\mathbf{G}_{n,m}$  from the center of the geodesic ball,  $C_0, a$  are both positive constants.) then

$$(\frac{\sqrt{2}}{4}\pi - \rho)^{-1} \leq 2C_0(|F|^2 + 2nt + 1)^a$$

for arbitrary  $a \geq 0$ .

Let  $M \rightarrow \mathbb{R}^4$  be a surface. Let  $\pi_1 : \mathbf{G}_{2,2} \rightarrow S^2$  be the projection of  $\mathbf{G}_{2,2}$  into its first factor, and  $\pi_2$  be the projection into the second factor. Define  $\gamma_i = \pi_i \circ \gamma$ . We also have

**Theorem 4.3.** *If the partial Gauss image of an initial surface  $M$  in  $\mathbb{R}^4$  is contained in a hemisphere, then the partial Gauss image of all the surfaces under MCF are same hemisphere.*

## 5. CURVATURE ESTIMATES

Let  $h : \mathbb{V} \rightarrow \mathbb{R}$  be a smooth function defined on an open subset  $\mathbb{V} \subset \mathbf{G}_{n,m}$ , and  $h \geq 1$ . Suppose that  $\text{Hess}(h)$  is nonnegative definite on  $\mathbb{V}$  and have the estimate

$$(5.1) \quad \text{Hess}(h) \geq 3hg + \frac{3}{2}h^{-1}dh \otimes dh,$$

where  $g$  is the metric tensor on  $\mathbf{G}_{n,m}$ .  $r$  is a smooth, non-negative function on  $\mathbb{R}^{n+m} \times \mathbb{R}$  satisfying

$$(5.2) \quad \left| \left( \frac{d}{dt} - \Delta \right) r \right| \leq C(n) \quad \text{and} \quad |\nabla r|^2 \leq C(n)r.$$

**Theorem 5.1.** *Let  $R > 0, T > 0$  be such that for any  $x \in \overline{M}_{t,R}$ , where  $t \in [0, T]$ , we have  $\gamma(x) \in \mathbb{V}$ . Then for any  $t \in [0, T]$  and  $\theta \in [0, 1)$ , we have the estimate*

$$\sup_{x \in \overline{M}_{t,\theta R}} |B|^2 \leq C(n)(1 - \theta^2)^{-2}(t^{-1} + R^{-2}) \sup_{x \in \overline{M}_{s,R}, s \in [0,t]} \tilde{h}^2,$$

where  $\tilde{h} = h \circ \gamma$ .

The proof of Theorem 5.1 shall be given later. At first we will see several applications of it.

Let  $r = |\mathbf{x}|^2$  for  $\mathbf{x} \in \mathbb{R}^n$ , then

$$\left| \left( \frac{d}{dt} - \Delta \right) r \right| = \left| 2x^i \left( \frac{d}{dt} - \Delta \right) x^i - 2|\nabla x^i|^2 \right| \leq 2n$$

and

$$|\nabla r|^2 = |2x^i \nabla x^i|^2 = 4(x^i)^2 |\nabla x^i|^2 \leq 4r.$$

Hence Theorem 5.1 yields

**Corollary 5.1.** *Let  $R > 0, T > 0$  be such that for any  $t \in [0, T]$ ,  $M_t \cap ((B_R \subset \mathbb{R}^n) \times \mathbb{R}^m)$  is a graph over  $B_R$ , i.e.  $M_t \cap ((B_R \subset \mathbb{R}^n) \times \mathbb{R}^m) = \{(x, f_t(x)) : x \in B_R\}$ , and  $\Delta_{f_t} < 2$ , then the following estimate holds for arbitrary  $t \in [0, T]$  and  $\theta \in [0, 1)$*

$$\sup_{(x, f_t(x)) \in K(t, \theta R)} |B|^2 \leq C(n)(1 - \theta^2)^{-2}(t^{-1} + R^{-2}) \sup_{s \in [0,t]} \sup_{(x, f_s(x)) \in K(s, R)} (2 - \Delta_{f_s})^{-3}.$$

Here

$$K(s, R) = \{(x, f_s(x)) : x \in B_R\}.$$

Combing Corollary 5.1 and Theorem 4.1 yields



**Corollary 5.2.** *If the initial submanifold is an entire graph over  $\mathbb{R}^n$ , i.e.  $M_0 = \text{graph} f_0$ , and  $\Delta_{f_0} < 2$ ,  $(2 - \Delta_{f_0})^{-1} = o(|x|^{2a})$ , then we have the estimate*

$$\sup_{(x, f_t(x)) \in K(t, \theta R)} |B|^2 \leq C(n)(1 - \theta^2)^{-2}(t^{-1} + R^{-2})(R^2 + 2nt + 1)^{3a}.$$

Here  $\theta \in [0, 1)$  and the denotation of  $K(\cdot, \cdot)$  is similar to Corollary 5.1.

Similarly, if

$$r = |\mathbf{x}|^2 + 2nt,$$

then it is easy to check that  $r$  satisfies (5.2). Applying Theorem 5.1 and Theorem 4.2 we have

**Corollary 5.3.** *Let  $R > 0, T > 0$  be such that for any  $t \in [0, T]$ , if  $x \in M_t$  satisfies  $|F|^2 + 2nt \leq R^2$ , then  $\gamma(x)$  lies in an open geodesic ball centered at a fixed point  $P_0$  of radius  $\frac{\sqrt{2}}{4}\pi$  in  $\mathbf{G}_{n,m}$ . Then the following estimate holds for arbitrary  $t \in [0, T]$  and  $\theta \in [0, 1)$*

$$\sup_{x \in K(t, \theta R)} |B|^2 \leq C(n)(1 - \theta^2)^{-2}t^{-1} \sup_{0 \leq s \leq t} \sup_{x \in K(s, R)} \left(\frac{\sqrt{2}}{4}\pi - \rho\right)^{-3}.$$

Here

$$K(s, R) = \{x \in M_s : |F|^2 + 2ns \leq R^2\}.$$

**Corollary 5.4.** *If the Gauss image of the initial complete submanifold  $M_0$  is contained in an open geodesic ball of radius  $\frac{\sqrt{2}}{4}\pi$  in  $\mathbf{G}_{n,m}$ , and  $(\frac{\sqrt{2}}{4}\pi - \rho)^{-1}$  has growth*

$$\left(\frac{\sqrt{2}}{4}\pi - \rho\right)^{-1} = o(|F|^{2a}),$$

then we have the estimate

$$\sup_{x \in K(t, \theta R)} |B|^2 \leq C(n)(1 - \theta^2)^{-2}t^{-1}(R^2 + 1)^{3a}.$$

Here  $\theta \in [0, 1)$  and the denotation of  $K(\cdot, \cdot)$  is similar to Corollary 5.3.

**Remark 5.1.** When  $x \in K(t, \theta R)$ ,

$$2nt \leq |F|^2 + 2nt \leq \theta^2 R^2 \leq R^2,$$

so

$$R^{-2} \leq \frac{1}{2n}t^{-1}.$$

Hence in the process of applying Theorem 5.1 to Corollary 5.3,  $t^{-1} + R^{-2}$  could be replaced by  $t^{-1}$ .

*Proof of Theorem 5.1.* Let  $\varphi = \varphi(\tilde{h})$  be a smooth nonnegative function of  $\tilde{h}$  to be determined later, and  $'$  denotes derivative with respect to  $\tilde{h}$ , then from (3.3), (3.5)

and (5.1) we have

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)|B|^2\varphi &= \left(\frac{d}{dt} - \Delta\right)|B|^2 \cdot \varphi + |B|^2\left(\frac{d}{dt} - \Delta\right)\varphi - 2\nabla|B|^2 \cdot \nabla\varphi \\
&\leq (-2|\nabla|B||^2 + 3|B|^4)\varphi \\
(5.3) \quad &+ |B|^2(\varphi'(\frac{d}{dt} - \Delta)\tilde{h} - \varphi''|\nabla\tilde{h}|^2) - 2\nabla|B|^2 \cdot \nabla\varphi \\
&\leq (-2|\nabla|B||^2 + 3|B|^4)\varphi - |B|^2\varphi'(3\tilde{h}|B|^2 + \frac{3}{2}\tilde{h}^{-1}|\nabla\tilde{h}|^2) \\
&\quad - |B|^2\varphi''|\nabla\tilde{h}|^2 - 2\nabla|B|^2 \cdot \nabla\varphi
\end{aligned}$$

The last term can be estimated by

$$\begin{aligned}
-2\nabla|B|^2 \cdot \nabla\varphi &= -\nabla|B|^2 \cdot \nabla\varphi - \nabla|B|^2 \cdot \nabla\varphi \\
&= -\varphi^{-1}(\nabla(|B|^2\varphi) - |B|^2\nabla\varphi) \cdot \nabla\varphi - 2|B|\nabla|B| \cdot \nabla\varphi \\
&\leq -\varphi^{-1}\nabla\varphi \cdot \nabla(|B|^2\varphi) + |B|^2\varphi^{-1}|\nabla\varphi|^2 \\
&\quad + 2|\nabla|B||^2\varphi + \frac{1}{2}|B|^2\varphi^{-1}|\nabla\varphi|^2 \\
(5.4) \quad &= -\varphi^{-1}\nabla\varphi \cdot \nabla(|B|^2\varphi) + 2|\nabla|B||^2\varphi + \frac{3}{2}|B|^2\varphi^{-1}|\nabla\varphi|^2.
\end{aligned}$$

Substituting (5.4) into (5.3) gives

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)|B|^2\varphi &\leq -(3\varphi'\tilde{h} - 3\varphi)|B|^4 \\
&\quad - \left(\frac{3}{2}\varphi'\tilde{h}^{-1} + \varphi'' - \frac{3}{2}\varphi^{-1}(\varphi')^2\right)|B|^2|\nabla\tilde{h}|^2 - \varphi^{-1}\nabla\varphi \cdot \nabla(|B|^2\varphi). \\
(5.5) \quad &
\end{aligned}$$

Now we let  $\varphi(\tilde{h}) = \frac{\tilde{h}}{1-k\tilde{h}}$ ,  $k \geq 0$  to be chosen; then

$$(5.6) \quad 3\varphi'\tilde{h} - 3\varphi = 3k\varphi^2,$$

$$(5.7) \quad \frac{3}{2}\varphi'\tilde{h}^{-1} + \varphi'' - \frac{3}{2}\varphi^{-1}(\varphi')^2 = \frac{k}{2\tilde{h}(1-k\tilde{h})^2}\varphi,$$

$$(5.8) \quad \varphi^{-1}\nabla\varphi = \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla\tilde{h}.$$

Substituting these identities into (5.5) we derive for  $g = |B|^2\varphi$  the inequality

$$(5.9) \quad \left(\frac{d}{dt} - \Delta\right)g \leq -3kg^2 - \frac{k}{2\tilde{h}(1-k\tilde{h})^2}|\nabla\tilde{h}|^2g - \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla\tilde{h} \cdot \nabla g.$$

As in Lemma 4.1, we define  $\eta = (R^2 - r)_+^2$ , then on the support of  $\eta$ ,

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\eta &= -2(R^2 - r)\left(\frac{d}{dt} - \Delta\right)r - 2|\nabla r|^2 \\
&\leq 2C(n)R^2 - 2|\nabla r|^2
\end{aligned}$$

and

$$\begin{aligned}
 \left(\frac{d}{dt} - \Delta\right)g\eta &= \left(\frac{d}{dt} - \Delta\right)g \cdot \eta + g\left(\frac{d}{dt} - \Delta\right)\eta - 2\nabla g \cdot \nabla \eta \\
 &\leq -3kg^2\eta - \frac{k}{2\tilde{h}(1-k\tilde{h})^2}|\nabla \tilde{h}|^2 g\eta - \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot \nabla g \cdot \eta \\
 (5.10) \quad &\quad + 2C(n)R^2g - 2g|\nabla r|^2 - 2\nabla g \cdot \nabla \eta;
 \end{aligned}$$

where

$$\begin{aligned}
 -2\nabla g \cdot \nabla \eta &= -2\eta^{-1}\nabla \eta \cdot \nabla(g\eta) + 2g\eta^{-1}|\nabla \eta|^2 \\
 (5.11) \quad &= -2\eta^{-1}\nabla \eta \cdot \nabla(g\eta) + 8g|\nabla r|^2
 \end{aligned}$$

and

$$\begin{aligned}
 &-\frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot \nabla g \cdot \eta \\
 &= -\frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot \nabla(g\eta) + \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot g\nabla \eta \\
 &\leq -\frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot \nabla(g\eta) + \frac{k}{2\tilde{h}(1-k\tilde{h})^2}|\nabla \tilde{h}|^2 g\eta + \frac{1}{2k\tilde{h}}g\eta^{-1}|\nabla \eta|^2 \\
 (5.12) \quad &= -\frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h} \cdot \nabla(g\eta) + \frac{k}{2\tilde{h}(1-k\tilde{h})^2}|\nabla \tilde{h}|^2 g\eta + \frac{2}{k\tilde{h}}g|\nabla r|^2.
 \end{aligned}$$

Substituting (5.11) and (5.12) into (5.10) gives

$$\begin{aligned}
 \left(\frac{d}{dt} - \Delta\right)g\eta &\leq -3kg^2\eta - \left(2\eta^{-1}\nabla \eta + \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h}\right) \cdot \nabla(g\eta) \\
 (5.13) \quad &\quad + C(n)\left[\left(1 + \frac{1}{k\tilde{h}}\right)r + R^2\right]g.
 \end{aligned}$$

Furthermore

$$\begin{aligned}
 \left(\frac{d}{dt} - \Delta\right)(tg\eta) &\leq -3ktg^2\eta - \left(2\eta^{-1}\nabla \eta + \frac{1}{\tilde{h}(1-k\tilde{h})}\nabla \tilde{h}\right) \cdot \nabla(tg\eta) \\
 (5.14) \quad &\quad + C(n)\left[\left(1 + \frac{1}{k\tilde{h}}\right)r + R^2\right]tg + g\eta.
 \end{aligned}$$

Denote

$$m(T) = \sup_{0 \leq t \leq T} \sup_{\overline{M}_{t,R}} tg\eta = t_0g(x_0, t_0)\eta(x_0, t_0),$$

then  $t_0 > 0$ ,  $r(x_0, t_0) < R^2$  and hence

$$\left(\frac{d}{dt} - \Delta\right)(tg\eta) \geq 0, \quad \nabla(tg\eta) = 0$$

at  $(x_0, t_0)$ . (5.14) implies

$$3kt_0g^2\eta \leq C(n)\left[\left(1 + \frac{1}{k\tilde{h}}\right)r + R^2\right]t_0g + g\eta.$$

Multiplying by  $\frac{t_0\eta}{3k}$  yields

$$\begin{aligned} m(T)^2 &\leq \frac{C(n)}{3k} \left(1 + \frac{1}{k\tilde{h}}\right) R^2 t_0^2 g\eta + \frac{t_0 g\eta^2}{3k} \\ &\leq \frac{C(n)}{3k} \left((1 + \frac{1}{k\tilde{h}}) R^2 T + \eta\right) m(T); \end{aligned}$$

by  $\eta = (R^2 - r)_+^2 \leq R^4$  we arrive at

$$g\eta T \leq m(T) \leq \frac{C(n)}{3k} \left((1 + \frac{1}{k\tilde{h}}) R^2 T + R^4\right)$$

in  $\overline{M}_{T,R}$ . Let now

$$(5.15) \quad k = \frac{1}{2} \inf_{x \in \overline{M}_{t,R}, t \in [0,T]} \tilde{h}^{-1}.$$

Since  $\varphi = \frac{\tilde{h}}{1-k\tilde{h}} \geq \frac{1}{1-k} \geq 1$  (by  $\tilde{h} \geq 1$ ) and  $\eta \geq (1 - \theta^2)^2 R^4$  in  $\overline{M}_{T,\theta R}$ , we have

$$(5.16) \quad \sup_{x \in \overline{M}_{T,\theta R}} |B|^2 \leq C(n) (1 - \theta^2)^{-2} (T^{-1} + R^{-2}) \sup_{t \in [0,T]} \sup_{x \in \overline{M}_{t,R}} \tilde{h}^2$$

and finally (5.1) follows from replacing  $T$  by  $t$ , replacing  $t$  by  $s$  in (5.16).  $\square$

Substituting  $\varphi = \tilde{h}$  into (5.5) gives

$$\left(\frac{d}{dt} - \Delta\right) |B|^2 \tilde{h} \leq -\tilde{h}^{-1} \nabla \tilde{h} \cdot \nabla (|B|^2 t dh).$$

Using the parabolic maximum principle for complete manifolds in [4], we have

**Corollary 5.5.** *Let  $M$  be a complete  $n$ -submanifold in  $\mathbb{R}^{m+n}$  with bounded curvature. Then*

$$\sup_{M_t} |B|^2 \tilde{h} \leq \sup_{M_0} |B|^2 \tilde{h}.$$

**Remark 5.2.** *When  $\mathbb{V}$  is a geodesic ball of radius  $\rho_0 < \frac{\sqrt{2}}{4}\pi$ , we can choose  $h = \sec^2(\sqrt{2}\rho)$ . So the above estimate is an improvement of Thm. 4.2 in [17].*

Furthermore, we can give a prior estimates for  $|\nabla^m B|^2$  by induction.

**Theorem 5.2.** *Our denotation and assumption is similar to Theorem 5.1, then for arbitrary  $m \geq 0$ ,  $\theta \in [0, 1)$  and  $t \in [0, T]$ , we have the estimate*

$$\sup_{x \in \overline{M}_{t,\theta R}} |\nabla^m B|^2 \leq c_m (R^{-2} + t^{-1})^{m+1};$$

where  $c_m = c_m(\theta, n, \sup_{\overline{M}_{s,R}, s \in [0,t]} \tilde{h})$ .

*Proof.* We proceed by induction on  $m$ . The case  $m = 0$  has been established as Theorem 5.1. Now we suppose the inequality holds for  $0 \leq k \leq m - 1$ . Denote  $\psi(t) = (R^{-2} + t^{-1})^{-1} = \frac{R^2 t}{R^2 + t}$ ; we shall estimate the upper bound of  $\psi^{m+1} |\nabla^m B|^2$  on  $\overline{M}_{T,\theta R}$  for fixed  $\theta \in [0, 1)$ .

By computing

$$(5.17) \quad \begin{aligned} \left(\frac{d}{dt} - \Delta\right)\psi^{m+1}|\nabla^m B|^2 &\leq -2\psi^{m+1}|\nabla^{m+1} B|^2 + \left(\frac{d}{dt}\psi^{m+1}\right)|\nabla^m B|^2 \\ &+ C(m, n)\psi^{m+1} \sum_{i+j+k=m, i \leq j \leq k} |\nabla^i B| |\nabla^j B| |\nabla^k B| |\nabla^m B|. \end{aligned}$$

By inductive assumption

$$\sup_{x \in \overline{M}_{t, \frac{1+\theta}{2}R}} \psi^{k+1} |\nabla^k B|^2 \leq c_k$$

for every  $0 \leq k \leq m-1$  and  $t \in [0, T]$ , where

$$c_k = c_k(\theta, n, \sup_{x \in \overline{M}_{t, R}, t \in [0, T]} \tilde{h})$$

(note that  $c_k$  depends on  $\frac{1+\theta}{2}$ , which only depends on  $\theta \in [0, 1)$ ); which implies  $|\nabla^i B| \leq c_i^{1/2} \psi^{-(i+1)/2}$ ,  $|\nabla^j B| \leq c_j^{1/2} \psi^{-(j+1)/2}$ ; moreover

$$(5.18) \quad \begin{aligned} &\psi^{m+1} \sum_{i+j+k=m, i \leq j \leq k} |\nabla^i B| |\nabla^j B| |\nabla^k B| |\nabla^m B| \\ &\leq C \sum_{i+j+k=m, i \leq j \leq k} \psi^{\frac{k+m}{2}} |\nabla^k B| |\nabla^m B| \\ &\leq C \sum_{k \leq m} \psi^k |\nabla^k B|^2. \end{aligned}$$

On the other hand

$$(5.19) \quad \frac{d}{dt} \psi^{m+1} = (m+1) \psi^m \frac{R^4}{(R^2 + t)^2} \leq (m+1) \psi^m.$$

Substituting (5.18) and (5.19) into (5.17) gives

$$(5.20) \quad \left(\frac{d}{dt} - \Delta\right)\psi^{m+1}|\nabla^m B|^2 \leq -2\psi^{m+1}|\nabla^{m+1} B|^2 + C \sum_{k \leq m} \psi^k |\nabla^k B|^2$$

on  $\overline{M}_{t, \frac{1+\theta}{2}R}$  for arbitrary  $t \in [0, T]$ ; where

$$C = C(\theta, n, \sup_{x \in \overline{M}_{t, R}, t \in [0, T]} \tilde{h}).$$

Now we define  $f = \psi^{m+1}|\nabla^m B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2)$ , where  $\Lambda > 0$  to be chosen later.

By computing

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right)f &\leq -2\psi^{m+1}|\nabla^{m+1} B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2) \\ &+ C \sum_{k \leq m} \psi^k |\nabla^k B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2) \\ &- 2\psi^{2m+1}|\nabla^m B|^4 + C \sum_{k \leq m-1} \psi^k |\nabla^k B|^2 \psi^{m+1} |\nabla^m B|^2 \end{aligned}$$

$$(5.21) \quad -2\psi^{2m+1}\nabla|\nabla^m B|^2 \cdot \nabla|\nabla^{m-1} B|^2;$$

where the last term can be estimated by

$$\begin{aligned}
& -2\psi^{2m+1}\nabla|\nabla^m B|^2 \cdot \nabla|\nabla^{m-1} B|^2 \\
& = -8\psi^{2m+1}|\nabla^m B|\nabla|\nabla^m B| \cdot |\nabla^{m-1} B|\nabla|\nabla^{m-1} B| \\
& \leq 2\psi^{m+1}|\nabla^{m+1} B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2) + 8\psi^{2m+1}|\nabla^m B|^4 \frac{\psi^m|\nabla^{m-1} B|^2}{\Lambda + \psi^m|\nabla^{m-1} B|^2} \\
(5.22) \quad & \leq 2\psi^{m+1}|\nabla^{m+1} B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2) + \frac{8c_{m-1}}{\Lambda + c_{m-1}}\psi^{2m+1}|\nabla^m B|^4.
\end{aligned}$$

Hence we derive

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)f & \leq -\left(2 - \frac{8c_{m-1}}{\Lambda + c_{m-1}}\right)\psi^{-1}(\psi^{m+1}|\nabla^m B|^2)^2 \\
& \quad + C\psi^{-1}\left(\sum_{k \leq m} \psi^{k+1}|\nabla^k B|^2(\Lambda + \psi^m|\nabla^{m-1} B|^2)\right. \\
(5.23) \quad & \quad \left. + \sum_{k \leq m-1} \psi^{k+1}|\nabla^k B|^2\psi^{m+1}|\nabla^m B|^2\right).
\end{aligned}$$

Now we let  $\Lambda = 7c_{m-1} + 1$ , then

$$\left(\frac{d}{dt} - \Delta\right)f \leq -\psi^{-1}(\Lambda + \psi^m|\nabla^{m-1} B|^2)^{-2}f^2 + C\psi^{-1}(1 + f);$$

by Young's inequality,

$$\begin{aligned}
Cf & \leq \frac{1}{2}(\Lambda + \psi^m|\nabla^{m-1} B|^2)^{-2}f^2 + \frac{1}{2}C^2(\Lambda + \psi^m|\nabla^{m-1} B|^2)^2 \\
& \leq \frac{1}{2}(\Lambda + \psi^m|\nabla^{m-1} B|^2)^{-2}f^2 + \frac{1}{2}C^2(8c_{m-1} + 1)^2;
\end{aligned}$$

hence we have

$$(5.24) \quad \left(\frac{d}{dt} - \Delta\right)f \leq -\psi^{-1}(\delta f^2 - C);$$

where

$$\delta = \frac{(C(8c_{m-1} + 1)^2 - 1)^2}{2(8c_{m-1} + 1)^2} > 0$$

and  $C$  is a positive constant depending on  $n, m, \sup_{\overline{M}_{t,R}, t \in [0, T]} \tilde{h}$ .

Now we let  $\varphi = \left(\frac{1+\theta}{2}R\right)^2 - r$ , and  $\eta = (\varphi_+)^2$ , then  $\eta$  is a nonnegative function which vanishes outside  $\overline{M}_{t, \frac{1+\theta}{2}R}$ ; similar to (5.10), we can derive

$$(5.25) \quad \left(\frac{d}{dt} - \Delta\right)f\eta \leq \psi^{-1}(\delta f^2 - C)\eta + C(n)R^2f - 2\eta^{-1}\nabla\eta \cdot \nabla(f\eta)$$

on  $\overline{M}_{t, \frac{1+\theta}{2}R}$ . Denote  $m(T) = \max_{0 \leq t \leq T} \max_{x \in \overline{M}_{t, \frac{1+\theta}{2}R}} f\eta = f\eta(x_0, t_0)$ , we have

$$f^2\eta \leq \frac{1}{\delta}(C\eta + C(n)R^2f\psi).$$

Multiplying by  $\eta$ , using  $\eta \leq R^4$ ,  $\psi \leq R^2$ , we have

$$\begin{aligned} f^2 \eta^2 &\leq \frac{1}{\delta} (C\eta^2 + C(n)R^2 f\eta\psi) \leq \frac{1}{\delta} (CR^8 + C(n)R^4 f\eta) \\ &\leq \frac{1}{\delta} \left( CR^8 + \frac{\delta}{2} f^2 \eta^2 + \frac{C(n)^2 R^8}{2\delta} \right) \end{aligned}$$

i.e.,  $m(T)^2 = f^2 \eta^2 \leq CR^8$ ,

$$\sup_{0 \leq t \leq T} \sup_{x \in \overline{M}_{t, \frac{1+\theta}{2}R}} f\eta \leq CR^4;$$

where  $C = C(\theta, n, m, \sup_{\overline{M}_{t,R}, t \in [0, T]} \tilde{h})$ .

Finally, since  $\eta = ((\frac{1+\theta}{2}R)^2 - (\theta R)^2)^2 = \frac{1+2\theta-3\theta^2}{4}R^4$  on  $\overline{M}_{T,R}$  and  $\Lambda + \psi^m |\nabla^{m-1} B|^2 \geq 7c_{m-1} + 1$ , we have

$$(5.26) \quad \sup_{x \in \overline{M}_{T, \theta R}} \psi^{m+1} |\nabla^m B|^2 \leq c_m(\theta, n, \sup_{x \in \overline{M}_{t,R}, t \in [0, T]} \tilde{h}).$$

and the conclusion follows from replacing  $T$  by  $t$  and replacing  $t$  by  $s$ .  $\square$

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